Chen-Ning Yang's conjecture of exterior product

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1 Introduction

Given a $\mathbb{R}$-vector space $V = \mathbb{R}^{2n}, n \in \mathbb{N}$, equipped with an inner product $\langle\ ,\ \rangle$, $\bigwedge^r V$ is the $\mathbb{R}$-vector space of degree r exterior forms. If $\{e_i\}_{i=1}^{2n}$ is an orthonormal basis of V, then $\{e_{i_1} \wedge e_{i_2} ... \wedge e_{i_r}, 1 \le i_1 < i_2 ... < i_r \le 2n\}$ is an orthonormal basis of $\bigwedge^r V$, $\dim \bigwedge^r V = \binom{2n}{r}$. For any $1 \le i_1 < i_2 ... < i_r \le 2n, 1 \le j_1 < j_2 ... < j_r \le 2n$,

$$\langle e_{i_1} \wedge e_{i_2} ... \wedge e_{i_r}, e_{j_1} \wedge e_{j_2} ... \wedge e_{j_r} \rangle = \delta_{i_1,j_1}\delta_{i_2,j_2}...\delta_{i_r,j_r},$$

For any $1 \le k, l \le n, k + l \le n$, $\xi \in \bigwedge^{2k} V, \eta \in \bigwedge^{2l} V$, find $\max_{\|\xi\|=\|\eta\|=1} \|\xi \wedge \eta\|$.

Conjecture 1 (Chen-Ning Yang)**.** The maximum is attained when

$$\xi_{max} = \frac{\omega^k}{\|\omega^k\|}, \quad \eta_{max} = \frac{\omega^l}{\|\omega^l\|}, \quad \omega = \sum_{i=1}^{n} e_{2i-1} \wedge e_{2i},$$

And we have

$$\omega^k = k! \sum_{1 \le i_1 < i_2 ... < i_k \le n} (e_{2i_1 - 1} \wedge e_{2i_1}) \wedge ... \wedge (e_{2i_k - 1} \wedge e_{2i_k})$$

Notice that $(e_1 \wedge e_2) \wedge (e_3 \wedge e_4) = (e_3 \wedge e_4) \wedge (e_1 \wedge e_2)$ has an even permutation on indices.

$$\|\omega^k\|_2^2 = k!^2 \binom{n}{k}, \quad \|\omega^l\|_2^2 = l!^2 \binom{n}{l}, \quad \|\omega^{k+l}\|_2^2 = (k+l)!^2 \binom{n}{k+l},$$

$$\|\xi_{max} \wedge \eta_{max}\|_2^2 = \frac{\|\omega^{k+l}\|_2^2}{\|\omega^k\|_2^2 \|\omega^l\|_2^2} = \frac{(k+l)!(n-k)!(n-l)!}{k!l!n!(n-k-l)!} \triangleq C(n,k,l),$$

$$C(n,k,l) = \frac{(k+l)!(l+m)!(k+m)!}{k!l!m!(k+l+m)!}, \quad m = n - k - l,$$

Let's start by analysing some special cases.

Example 1. $k + l = n$, now $\bigwedge^{2k+2l} V = \bigwedge^{2n} V = \mathbb{R} e_1 \wedge e_2 \wedge ... e_{2n-1} \wedge e_{2n}$. Assume that $u = \sum_I u_I e_I \in \bigwedge^{2k} V$, $v = \sum_J v_J e_J \in \bigwedge^{2l} V$, where

$$I = (i_1, i_2, ..., i_{2k}), \quad 1 \le i_1 < i_2 ... < i_{2k} \le 2n, \quad e_I = e_{i_1} \wedge e_{i_2} ... \wedge e_{i_{2k}},$$

$$J = (j_1, j_2, ..., j_{2l}), \quad 1 \le j_1 < j_2 ... < j_{2l} \le 2n, \quad e_J = e_{j_1} \wedge e_{j_2} ... \wedge e_{j_{2l}},$$

then we have

$$u \wedge v = \sum_{I \cap J = \emptyset} u_I v_J e_I \wedge e_J = \sum_{I \cap J = \emptyset} u_I v_J \mathrm{sgn}(\sigma_{I,J}) e_{[2n]},$$

$$\sigma_{I,J} : (I, J) \to [2n] = (1, 2, ..., 2n), \quad e_{[2n]} = e_1 \wedge e_2 \wedge ...e_{2n-1} \wedge e_{2n},$$

$\mathrm{sgn}(\sigma_{I,J})$ is the signature of permutation $\sigma_{I,J}$: $\mathrm{sgn}(\sigma_{I,J}) = 1$ if $\sigma_{I,J}$ is an even permutation, $\mathrm{sgn}(\sigma_{I,J}) = -1$ if $\sigma_{I,J}$ is an odd permutation.

$$\|u \wedge v\|_2 = |\sum_{I \cap J = \emptyset} u_I v_J \mathrm{sgn}(\sigma_{I,J})| \leq \|u\|_2 \|v\|_2,$$

holds by Cauchy's inequality, since for any given I, $J = [2n] \backslash I$ is uniquely determined. Since $C(n, k, l) = 1$ in this case, the conjecture holds. Notice that in this case, for any given $u \in \bigwedge^{2k} V$ with $\|u\| = 1$, there exists $v \in \bigwedge^{2l} V$ with $\|v\| = 1$ such that the equality $\|u \wedge v\| = \|u\| \|v\|$ holds.

Property 1 (Construction of exterior product by tensor product and quotient map)**.** $\wedge : \bigwedge^{2k} V \times \bigwedge^{2l} V \to \bigwedge^{2k+2l} V$ is a bilinear map:

$$(\xi_1 + \xi_2) \wedge \eta = \xi_1 \wedge \eta + \xi_2 \wedge \eta, \quad \xi \wedge (\eta_1 + \eta_2) = \xi \wedge \eta_1 + \xi \wedge \eta_2,$$

$$(a\xi) \wedge \eta = a(\xi \wedge \eta) = \xi \wedge (a\eta), \quad a \in \mathbb{R},$$

So we have the universal property of tensor product:

$$\begin{array}{ccc} \bigwedge^{2k} V \times \bigwedge^{2l} V & \xrightarrow{\varphi} & \bigwedge^{2k} V \otimes \bigwedge^{2l} V \\ & {\searrow}^{\wedge} & \downarrow \sim \\ & & \bigwedge^{2k+2l} V \end{array}$$

where $\varphi : \bigwedge^{2k} V \times \bigwedge^{2l} V \to \bigwedge^{2k} V \otimes \bigwedge^{2l} V, \quad (u, v) \mapsto u \otimes v$ is the canonical bilinear map, and $\sim$: $\bigwedge^{2k} V \otimes \bigwedge^{2l} V \to \bigwedge^{2k+2l} V$ is the quotient map defined as follows: for $I \subset [2n], |I| = 2k, J \subset [2n], |J| = 2l$, if $I \cap J = \emptyset$, then let

$$\sim (e_I \otimes e_J) = \mathrm{sgn}(\sigma_{I,J}) e_{I \cup J}, \quad \sigma_{I,J} : (I, J) \to I \cup J,$$

where $I, J, I \cup J$ are sorted in ascending order, $\mathrm{sgn}(\sigma_{I,J})$ is the signature of permutation $\sigma_{I,J}$. If $I \cap J \neq \emptyset$, then let $\sim (e_I \otimes e_J) = 0$.

Property 2 (Universal property of exterior product)**.** Assume $V = \mathbb{R}^n$, the space of degree m exterior forms on V is $\bigwedge^m V$. It has the following universal property: if $\varphi : V^{\times m} \to W$ is a m-linear antisymmetric (alternating) map where $V^{\times m}$ is the product of m copies of V and W is a $\mathbb{R}$-vector space, then there exists a unique m-linear antisymmetric map $\tilde{\varphi}$ such that the following diagram commutes:

$$\begin{array}{ccc} V^{\times m} & \longrightarrow & \bigwedge^m V \\ & {\searrow}^{\varphi} & \downarrow \tilde{\varphi} \\ & & W \end{array}$$

Especially, when $m = 1$, $\bigwedge^1 V$ is the dual vector space V^* of V.

Property 3. Monotonicity of $C(n, k, l)$: for fixed $k, l \in \mathbb{N}_+$, $n \geq k + l$, we have

$$C(n + 1, k, l) > C(n, k, l)... > C(k + l, k, l) = 1,$$

Proof. Let $m = n + 1 - k - l$, since $\frac{(m+l)(m+k)}{m(m+k+l)} > 1$, we have $C(n + 1, k, l) > C(n, k, l)$. □

Property 4. If there exists a subspace $V_1 \subset V$ with $\dim V_1 = 2k$, such that $\xi \in \bigwedge^{2k} V$ is the unit volume form of V_1, and there exists a subspace $V_2 \subset V$ with $\dim V_2 = 2l$, such that $\eta \in \bigwedge^{2l} V$ is the unit volume form of V_2. Then $\|\xi\| = \|\eta\| = 1$, and V_1 has an orthonormal basis $\{e_i\}_{i=1}^{2k}$, such that $\xi = e_1 \wedge e_2 ... \wedge e_{2k}$, V_2 has an orthonormal basis $\{e'_j\}_{j=1}^{2l}$, such that $\eta = e'_1 \wedge e'_2 ... \wedge e'_{2l}$. Then by Hadamard's inequality, we have

$$\|\xi \wedge \eta\| = \|e_1 \wedge e_2 ... \wedge e_{2k} \wedge e'_1 \wedge e'_2 ... \wedge e'_{2l}\| \leq \prod_{i=1}^{2k} \|e_i\| \prod_{j=1}^{2l} \|e'_j\| = 1,$$

If U, V are Hilbert spaces on $\mathbb{R}$ with inner products $\langle \ , \ \rangle_U$ and $\langle \ , \ \rangle_V$, we may construct an inner product on $U \otimes V$ as follows:

$$\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle_{U \otimes V} = \langle u_1, u_2 \rangle_U \langle v_1, v_2 \rangle_V,$$

Notice that under this inner product on $U \otimes V$, the canonical bilinear map $\varphi : U \times V \to U \otimes V$ is an isometry onto $\mathrm{im}\varphi$ in the following sense: $\|u \otimes v\|_{U \otimes V} = \|u\|_U \|v\|_V$. φ is a Segre embedding, $\mathrm{im}\varphi$ is a Segre variety. If $\{e_i\}_{i=1}^m$ is an orthonormal basis of U, $\{e'_j\}_{j=1}^n$ is an orthonormal basis of V, then $\{e_i \otimes e'_j\} 1 \leq i \leq m, 1 \leq j \leq n$ is an orthonormal basis of $U \otimes V$. The exact form of φ is as follows:

$$u = \sum_{i=1}^m u_i e_i, \quad v = \sum_{j=1}^n v_j e'_j, \quad \varphi(u, v) = u \otimes v = \sum_{1 \leq i \leq m, 1 \leq j \leq n} u_i v_j e_i \otimes e'_j,$$

and we have $\dim \mathrm{im}\varphi = \dim U + \dim V - 1$. In the case of Yang's conjecture, we have

$$\dim \bigwedge^{2k} V = \binom{2n}{2k}, \quad \dim \bigwedge^{2l} V = \binom{2n}{2l}, \quad \dim \bigwedge^{2k+2l} V = \binom{2n}{2k+2l},$$

$$\dim \bigwedge^{2k} V \otimes \bigwedge^{2l} V = \binom{2n}{2k}\binom{2n}{2l}, \quad \dim \mathrm{im}\varphi = \binom{2n}{2k} + \binom{2n}{2l} - 1,$$

Example 2. $V = \mathbb{R}^2$, $u, v \in \bigwedge^1 V$, $u = u_1 e_1 + u_2 e_2$, $v = v_1 e_1 + v_2 e_2$, $u \wedge v = (u_1 v_2 - u_2 v_1) e_{12}$,

$$\frac{\|u \wedge v\|^2}{\|u\|^2 \|v\|^2} = \frac{(u_1 v_2 - u_2 v_1)^2}{(u_1^2 + u_2^2)(v_1^2 + v_2^2)} \leq 1,$$

If $u \otimes v = a_{11} e_1 \otimes e_1 + a_{12} e_1 \otimes e_2 + a_{21} e_2 \otimes e_1 + a_{22} e_2 \otimes e_2$, then $a_{11} a_{22} = a_{12} a_{21}$. If we remove the condition $a_{11} a_{22} = a_{12} a_{21}$ and consider arbitrary $w \in \bigwedge^1 V \otimes \bigwedge^1 V$, $w = a e_1 \otimes e_1 + b e_1 \otimes e_2 + c e_2 \otimes e_1 + d e_2 \otimes e_2$, then we have

$$\frac{\| \sim (w)\|^2}{\|w\|^2} = \frac{(b - c)^2}{a^2 + b^2 + c^2 + d^2} \leq 2,$$

Notice that the upper bound becomes loose. This example shows that we cannot only consider the quotient map $\sim : \bigwedge^1 V \otimes \bigwedge^1 V \to \bigwedge^2 V$ on the whole space $\bigwedge^1 V \otimes \bigwedge^1 V$, we must consider its restriction on the Segre variety $\mathrm{im}\varphi$. In the above case, if we add the constraint $ad = bc$, then we have

$$\frac{(b - c)^2}{a^2 + b^2 + c^2 + d^2} \leq \frac{(b - c)^2 + (a + d)^2}{a^2 + b^2 + c^2 + d^2} = 1,$$

Assume that $u = \sum_I u_I e_I \in \bigwedge^{2k} V$, $v = \sum_J v_J e_J \in \bigwedge^{2l} V$, then we have

$$u \wedge v = \sum_{I \cap J = \emptyset} u_I v_J e_I \wedge e_J = \sum_{|P| = 2k+2l} e_P \sum_{I \cup J = P} u_I v_J \mathrm{sgn}(\sigma_{I,J}),$$

$$\|u \wedge v\|^2 = \sum_{|P| = 2k+2l} \Big(\sum_{I \cup J = P} u_I v_J \mathrm{sgn}(\sigma_{I,J}) \Big)^2 = \sum_{|P| = 2k+2l,\ I_1 \cup J_1 = I_2 \cup J_2 = P} u_{I_1} v_{J_1} u_{I_2} v_{J_2} \mathrm{sgn}(\sigma_{I_1,J_1}) \mathrm{sgn}(\sigma_{I_2,J_2}),$$

Example 3 (Volume form of a linear subspace)**.** Assume $V=\mathbb{R}^4$, $\xi\in\bigwedge^2 V$, and there exists a subspace $V_1\subset V$, $\dim V_1=2$, such that $\xi=e_1'\wedge e_2'$ where $\{e_1',e_2'\}$ is an orthonormal basis of V_1. Assume that $e_j'=\sum_{i=1}^4 a_{ij}e_i$, then we have

$$(e_1'\ e_2')=(e_1\ e_2\ e_3\ e_4)\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\\a_{31}&a_{32}\\a_{41}&a_{42}\end{pmatrix}=(e_1\ e_2\ e_3\ e_4)A,\quad A^tA=I_2,$$

$$e_1'\wedge e_2'=\sum_{i<j}\det\begin{pmatrix}a_{i1}&a_{i2}\\a_{j1}&a_{j2}\end{pmatrix}e_i\wedge e_j,$$

More generally, assume $V=\mathbb{R}^n$, $\xi\in\bigwedge^m V$, and there exists a set of orthonormal vectors $\{e_j'\}_{j=1}^m$ such that $\xi=e_1'\wedge e_2'...\wedge e_m'$. Assume that $e_j'=\sum_{i=1}^n a_{ij}e_i$, then we have

$$(e_1'\ e_2'...\ e_m')=(e_1\ e_2...\ e_n)A,\quad A\in M(n,m,\mathbb{R}),\quad A^tA=I_m,$$

$$e_1'\wedge e_2'...\wedge e_m'=\sum_{|I|=m}\det A_{I,[m]}e_I,\quad I=(i_1,i_2,...i_m),\quad 1\le i_1<i_2...<i_m\le n,$$

Example 4. Assume $V=\mathbb{C}^G=L^2(G)$, $G=\mathbb{Z}/2n\mathbb{Z}$, $f,g\in V$. I tried to adopt the theory of Fourier analysis on finite abelian groups but failed.

Proposition 1 (Decomposition of 2-forms)**.** Assume $V=\mathbb{R}^{2n}$, $\xi\in\bigwedge^2 V$, $\|\xi\|=1$, then there exists a basis $\{e_i'\}_{i=1}^{2n}$ of V such that

$$\xi=\sum_{i=1}^n a_ie_{2i-1}'\wedge e_{2i}',\quad \sum_{i=1}^n a_i^2=1,\quad a_1\ge a_2...\ge a_n\ge 0,$$

Especially, when $n=2$, we can write $\xi=\cos\theta e_1'\wedge e_2'+\sin\theta e_3'\wedge e_4'$; when $n=3$, we can write $\xi=\cos\theta e_1'\wedge e_2'+\sin\theta\cos\phi e_3'\wedge e_4'+\sin\theta\sin\phi e_5'\wedge e_6'$.

Proof. This is equivalent to the fact that a real skew-symmetric matrix $A\in M(2n,2n,\mathbb{R})$ can be written in the form $A=Q^t\Sigma Q$, where $Q\in SO(2n)$ and Σ is the real canonical form.

$$a_1=\max_{e_1',e_2'}\langle\xi,e_1'\wedge e_2'\rangle,$$

$a_2,...a_n$ can be determined recursively. □

Example 5. $n=3$, $V=\mathbb{R}^6$, $k=l=1$, $C(n,k,l)=\frac{2!2!2!}{1!1!1!3!}=\frac{4}{3}$. For each $P\subset[2n]$, $|P|=2k+2l=4$, the bilinear form $\sum_{I\cup J=P}u_Iv_J\mathrm{sgn}(\sigma_{I,J})$ contains $\binom{2k+2l}{2k}=6$ terms. But if we use the decomposition of 2-forms, for a given $\xi\in\bigwedge^2 V$, $\|\xi\|=1$, there exists a basis $\{e_i\}_{i=1}^{2n}$ such that we can write

$$\xi=a_1e_1\wedge e_2+a_2e_3\wedge e_4+a_3e_5\wedge e_6,\quad a_1^2+a_2^2+a_3^2=1,$$

i) If $\eta=\sum_{i=1}^n b_ie_{2i-1}\wedge e_{2i}$, $\sum_{i=1}^n b_i^2=1$, then

$$\xi\wedge\eta=\sum_{1\le i<j\le n}(a_ib_j+a_jb_i)e_{2i-1}\wedge e_{2i}\wedge e_{2j-1}\wedge e_{2j},$$

$$\|\xi\wedge\eta\|^2=\sum_{1\le i<j\le n}(a_ib_j+a_jb_i)^2=(a_1b_2+a_2b_1)^2+(a_1b_3+a_3b_1)^2+(a_2b_3+a_3b_2)^2,$$

$$\frac{4}{3}\|\xi\|^2\|\eta\|^2 - \|\xi\wedge\eta\|^2 = \frac{4}{3}(a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2) + \frac{1}{3}\sum_{i<j}(a_i^2b_j^2 + a_j^2b_i^2) - 2\sum_{i<j} a_ib_ja_jb_i$$
$$= \frac{1}{3}\sum_{i<j}(a_ib_j - a_jb_i)^2 + \frac{2}{3}\sum_{i<j}(a_ib_i - a_jb_j)^2 \geq 0,$$

ii) If $\eta = \sum_{i<j} b_{ij}e_i\wedge e_j$, with $b_{2i-1,2i} = 0, 1\leq i\leq n$, then η lies in the orthogonal complement of $\bigoplus_{i=1}^n \mathbb{R}e_{2i-1}\wedge e_{2i}$ in $\bigwedge^2 V$. Then we have

$$\|\xi\wedge\eta\|^2 = a_3^2(b_{13}^2 + b_{14}^2 + b_{23}^2 + b_{24}^2) + a_2^2(b_{15}^2 + b_{16}^2 + b_{25}^2 + b_{26}^2) + a_1^2(b_{35}^2 + b_{36}^2 + b_{45}^2 + b_{46}^2),$$

So $\|\xi\wedge\eta\|^2 \leq \|\xi\|^2\|\eta\|^2 \leq \frac{4}{3}\|\xi\|^2\|\eta\|^2$ holds.

iii) Consider arbitrary $\eta = \sum_{i<j} b_{ij}e_i\wedge e_j$, we prove $\|\xi\wedge\eta\|^2 \leq \frac{4}{3}\|\xi\|^2\|\eta\|^2$ using sum of squares.

$$\frac{4}{3}\|\xi\|^2\|\eta\|^2 - \|\xi\wedge\eta\|^2 = \frac{4}{3}(a_1^2 + a_2^2 + a_3^2)\sum_{i<j} b_{ij}^2 - a_1^2(b_{34}^2 + b_{35}^2 + b_{36}^2 + b_{45}^2 + b_{46}^2 + b_{56}^2)$$
$$- a_2^2(b_{12}^2 + b_{15}^2 + b_{16}^2 + b_{25}^2 + b_{26}^2 + b_{56}^2) - a_3^2(b_{12}^2 + b_{12}^2 + b_{14}^2 + b_{23}^2 + b_{24}^2 + b_{34}^2)$$
$$- 2a_1a_2b_{12}b_{34} - 2a_1a_3b_{12}b_{56} - 2a_2a_3b_{34}b_{56}$$
$$= \frac{4}{3}a_1^2(b_{12}^2 + b_{13}^2 + b_{14}^2 + b_{15}^2 + b_{16}^2 + b_{23}^2 + b_{24}^2 + b_{25}^2 + b_{26}^2) + \frac{4}{3}a_2^2(b_{34}^2 + b_{13}^2 + b_{24}^2 + b_{35}^2$$
$$+ b_{36}^2 + b_{14}^2 + b_{24}^2 + b_{45}^2 + b_{46}^2) + \frac{4}{3}a_3^2(b_{56}^2 + b_{15}^2 + b_{25}^2 + b_{35}^2 + b_{45}^2 + b_{16}^2 + b_{26}^2 + b_{36}^2 + b_{46}^2)$$
$$+ \frac{1}{3}a_1^2(b_{34}^2 + b_{35}^2 + b_{36}^2 + b_{45}^2 + b_{46}^2 + b_{56}^2) + \frac{1}{3}a_2^2(b_{12}^2 + b_{15}^2 + b_{16}^2 + b_{25}^2 + b_{26}^2 + b_{56}^2)$$
$$+ \frac{1}{3}a_3^2(b_{12}^2 + b_{12}^2 + b_{14}^2 + b_{23}^2 + b_{24}^2 + b_{34}^2) - 2a_1a_2b_{12}b_{34} - 2a_1a_3b_{12}b_{56} - 2a_2a_3b_{34}b_{56}$$
$$= \frac{2}{3}(a_1b_{12} - a_2b_{34})^2 + \frac{2}{3}(a_1b_{12} - a_3b_{56})^2 + \frac{2}{3}(a_2b_{34} - a_3b_{56})^2 + \frac{1}{3}(a_1b_{34} - a_2b_{12})^2$$
$$+ \frac{1}{3}(a_1b_{56} - a_3b_{12})^2 + \frac{1}{3}(a_2b_{56} - a_3b_{34})^2 + remainders \geq 0,$$

Example 6. $n\geq 3$, $V = \mathbb{R}^{2n}$, $k = l = 1$, $C(n,k,l) = \frac{(n-1)!(n-1)!2!}{1!1!(n-2)!n!} = \frac{2(n-1)}{n}$. Using the decomposition of 2-forms, for a given $\xi\in\bigwedge^2 V$, $\|\xi\| = 1$, there exists a basis $\{e_i\}_{i=1}^{2n}$ such that we can write

$$\xi = \sum_{i=1}^n a_ie_{2i-1}\wedge e_{2i}, \quad \sum_{i=1}^n a_i^2 = 1, \quad a_1\geq a_2 ...\geq a_n\geq 0,$$

i) If $\eta = \sum_{i=1}^n b_ie_{2i-1}\wedge e_{2i}$, $\sum_{i=1}^n b_i^2 = 1$, then

$$\xi\wedge\eta = \sum_{1\leq i<j\leq n}(a_ib_j + a_jb_i)e_{2i-1}\wedge e_{2i}\wedge e_{2j-1}\wedge e_{2j},$$

$$\frac{2(n-1)}{n}\|\xi\|^2\|\eta\|^2 - \|\xi\wedge\eta\|^2 = \frac{2(n-1)}{n}(\sum_{i=1}^n a_i^2)(\sum_{i=1}^n b_i^2) - \sum_{1\leq i<j\leq n}(a_ib_j + a_jb_i)^2$$
$$= \frac{2(n-1)}{n}(\sum_{i=1}^n a_i^2b_i^2) + \frac{n-2}{n}\sum_{i<j}(a_i^2b_j^2 + a_j^2b_i^2) - 2\sum_{i<j} a_ib_ja_jb_i$$
$$= \frac{n-2}{n}\sum_{i<j}(a_ib_j - a_jb_i)^2 + \frac{2}{n}\sum_{i<j}(a_ib_i - a_jb_j)^2 \geq 0,$$

ii) If $\eta = \sum_{i<j} b_{ij} e_i \wedge e_j$, with $b_{2i-1,2i} = 0, 1 \le i \le n$, then

$$\|\xi \wedge \eta\|^2 = \sum_{i=1}^{n} a_i^2 \sum_{J \neq (i,j)} b_J^2 \le \|\xi\|^2 \|\eta\|^2 \le \frac{2(n-1)}{n} \|\xi\|^2 \|\eta\|^2,$$

iii) For arbitrary $\eta = \sum_{i<j} b_{ij} e_i \wedge e_j$, we prove $\|\xi \wedge \eta\|^2 \le \frac{2(n-1)}{n} \|\xi\|^2 \|\eta\|^2$ using sum of squares.

$$\frac{2(n-1)}{n} \|\xi\|^2 \|\eta\|^2 - \|\xi \wedge \eta\|^2 = \frac{2(n-1)}{n} \sum_{k=1}^{n} a_k^2 \sum_{i<j} b_{ij}^2 - \sum_{k=1}^{n} a_k^2 (\sum_{2k-1,2k \notin J} b_J^2) - 2 \sum_{k<l} a_k a_l b_{2k-1,2k} b_{2l-1,2l}$$

$$= \frac{2(n-1)}{n} \sum_{k=1}^{n} a_k^2 (\sum_{2k-1 \text{ or } 2k \in J} b_J^2) + \frac{n-2}{n} \sum_{k=1}^{n} a_k^2 (\sum_{2k-1,2k \notin J} b_J^2) - 2 \sum_{k<l} a_k a_l b_{2k-1,2k} b_{2l-1,2l}$$

$$= \frac{n-2}{n} \sum_{k<l} (a_k b_{2l-1,2l} - a_l b_{2k-1,2k})^2 + \frac{2}{n} \sum_{k<l} (a_k b_{2k-1,2k} - a_l b_{2l-1,2l})^2 + remainders \ge 0,$$

So we finished proving the case when $k = l = 1$.

Example 7. $k = 1, l \ge 2, n \ge l+1$, $C(n,k,l) = \frac{(l+1)!(n-1)!(n-l)!}{1!l!n!(n-1-l)!} = \frac{(l+1)(n-l)}{n}$. Using the decomposition of 2-forms, for a given $\xi \in \bigwedge^2 V$, $\|\xi\| = 1$, there exists a basis $\{e_i\}_{i=1}^{2n}$ such that we can write

$$\xi = \sum_{i=1}^{n} a_i e_{2i-1} \wedge e_{2i}, \quad \sum_{i=1}^{n} a_i^2 = 1, \quad a_1 \ge a_2 ... \ge a_n \ge 0,$$

1) We first consider the case when $l = 2$, now $C(n,k,l) = \frac{3(n-2)}{n}$.
i) If $\eta = \sum_{1 \le i<j \le n} b_{ij} e_{2i-1} \wedge e_{2i} \wedge e_{2j-1} \wedge e_{2j}$, $\sum_{1 \le i<j \le n} b_{ij}^2 = 1$, then

$$\xi \wedge \eta = \sum_{p<i<j} (a_p b_{ij} + a_i b_{pj} + a_j b_{pi}) e_{\{2p-1,2p,2i-1,2i,2j-1,2j\}},$$

$$\frac{3(n-2)}{n} \|\xi\|^2 \|\eta\|^2 - \|\xi \wedge \eta\|^2 = \frac{3(n-2)}{n} (\sum_{p} a_p^2)(\sum_{i<j} b_{ij}^2) - \sum_{p<i<j} (a_p b_{ij} + a_i b_{pj} + a_j b_{pi})^2$$

$$= \frac{3(n-2)}{n} \sum_{i<j} (a_i^2 + a_j^2) b_{ij}^2 + \frac{2(n-3)}{n} \sum_{p<i<j} (a_p^2 b_{ij}^2 + a_i^2 b_{pj}^2 + a_j^2 b_{pi}^2)$$
$$- 2 \sum_{p<i<j} (a_p b_{ij} a_i b_{pj} + a_p b_{ij} a_j b_{pi} + a_i b_{pj} a_j b_{pi})$$

$$= \frac{2(n-3)}{n} \sum_{p<i<j} (a_p^2 b_{ij}^2 + a_i^2 b_{pj}^2 + a_j^2 b_{pi}^2 - a_p b_{ij} a_i b_{pj} - a_p b_{ij} a_j b_{pi} - a_i b_{pj} a_j b_{pi})$$
$$+ \frac{3}{n} \sum_{p<i<j} ((a_p^2 + a_i^2) b_{pi}^2 + (a_p^2 + a_j^2) b_{pj}^2 + (a_i^2 + a_j^2) b_{ij}^2 - 2(a_p b_{ij} a_i b_{pj} + a_p b_{ij} a_j b_{pi} + a_i b_{pj} a_j b_{pi})) \ge 0,$$

ii) For arbitrary $\eta = \sum_{J \subset [2n], |J|=4} b_J e_J$, $\sum_{J \subset [2n], |J|=4} b_J^2 = 1$,

$$\|\xi \wedge \eta\|^2 = \sum_{I \cap J = \emptyset} a_I^2 b_J^2 + \sum_{I_1 \neq I_2} a_{I_1} a_{I_2} b_{J_1} b_{J_2}, \quad J_1 = I_2 \cup K, \quad J_2 = I_1 \cup K,$$

where $|K| = 2, K \cap I_1 = K \cap I_2 = \emptyset$, and we used the following equality:

$$\text{sgn}(I_1, J_1)\text{sgn}(I_2, J_2) = \text{sgn}(I_1, I_2, K)\text{sgn}(I_2, K)\text{sgn}(I_2, I_1, K)\text{sgn}(I_1, K) = 1,$$

$$\frac{3(n-2)}{n}\|\xi\|^2\|\eta\|^2-\|\xi\wedge\eta\|^2=\frac{3(n-2)}{n}\sum_{I\cap J\neq\emptyset}a_I^2b_J^2+\frac{2(n-3)}{n}\sum_{I\cap J=\emptyset}a_I^2b_J^2-\sum_{I_1\neq I_2}a_{I_1}a_{I_2}b_{J_1}b_{J_2}$$

$$=\frac{3}{n}\sum_{I_1\neq I_2}(\frac{1}{2}(a_{I_1}^2b_{J_2}^2+a_{I_2}^2b_{J_1}^2)-a_{I_1}a_{I_2}b_{J_1}b_{J_2})+\frac{n-3}{n}\sum_{I_1\neq I_2}(\frac{1}{2}(a_{I_1}^2b_{J_1}^2+a_{I_2}^2b_{J_2}^2)-a_{I_1}a_{I_2}b_{J_1}b_{J_2})$$

$$+\,remainders$$

2) For arbitrary $l\geq 2$, now $C(n,k,l)=\frac{(l+1)(n-l)}{n}$.

i) If $\eta=\sum_{|I|=l,\,I\subset[n]}b_Ie_{\{2i-1,2i,\,i\in I\}}$, $\sum_{|I|=l,\,I\subset[n]}b_I^2=1$, then

$$\xi\wedge\eta=\sum_{|J|=l+1,\,J\subset[n]}(\sum_{i\in J}a_ib_{J\backslash\{i\}})e_{\{2j-1,2j,\,j\in J\}},$$

$$\frac{(l+1)(n-l)}{n}\|\xi\|^2\|\eta\|^2-\|\xi\wedge\eta\|^2=\frac{(l+1)(n-l)}{n}(\sum_p a_p^2)(\sum_{|I|=l,\,I\subset[n]}b_I^2)-\sum_{|J|=l+1,\,J\subset[n]}(\sum_{i\in J}a_ib_{J\backslash\{i\}})^2$$

$$=\frac{(l+1)(n-l)}{n}\sum_{|I|=l,\,I\subset[n]}(\sum_{i\in I}a_i^2)b_I^2+\frac{l(n-l-1)}{n}\sum_{|J|=l+1,\,J\subset[n]}(\sum_{i\in J}a_i^2b_{J\backslash\{i\}}^2)$$

$$-2\sum_{|J|=l+1,\,J\subset[n]}(\sum_{i<j\in J}a_ib_{J\backslash\{i\}}a_jb_{J\backslash\{j\}})$$

$$=\frac{n-1-l}{n}\sum_{|J|=l+1,\,J\subset[n]}(l\sum_{i\in J}a_i^2b_{J\backslash\{i\}}^2-2\sum_{i<j\in J}a_ib_{J\backslash\{i\}}a_jb_{J\backslash\{j\}})$$

$$+\frac{l+1}{n}\sum_{|J|=l+1,\,J\subset[n]}(\sum_{i\in J}(\sum_{j\in J\backslash\{i\}}a_j^2)b_{J\backslash\{i\}}^2-2\sum_{i<j\in J}a_ib_{J\backslash\{i\}}a_jb_{J\backslash\{j\}})\geq 0,$$

ii) For arbitrary $\eta=\sum_{J\subset[2n],|J|=2l}b_Je_J$, $\sum_{J\subset[2n],|J|=2l}b_J^2=1$,

Conjecture 2 (Birch)**.** The only remaining case is when $n=5$, $z_1,z_2,...z_n\in\mathbb{C}$ satisfy the constraint $\sum_{i=1}^n|z_i|^2=n$, then

$$\Delta=\prod_{1\leq i<j\leq n}|z_i-z_j|^2\leq n^n,$$

The equality holds when $z_1,z_2,...z_n$ are the vertices of a pentagon on the unit circle.

$$f(z)=\prod_{i=1}^n(z-z_i),\quad \Delta=\prod_{i=1}^n|f'(z_i)|,\quad f_{max}(z)=z^n-c,\quad |f'_{max}(z_i)|=|nz_i^{n-1}|=n,$$

References

[1] Kailiang Lin, tba.